L'integrazione di una funzione razionale fratta è sempre ricondotta all'integrazione di una funzione razionale fratta propria *(una frazione in cui il grado del numeratore è minore del grado del denominatore).*

Infatti se la frazione è impropria, effettuando la divisione fra il numeratore e il denominatore, si ottiene la somma di una funzione razionale intera e una funzione razionale fratta propria:

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Per integrare una funzione razionale fratta propria del tipo $\int \frac{N(x)}{D(x)}\,dx$ occorre distinguere i seguenti casi:

I° CASO – $D(x) = ax + b$

$$\int \frac{k}{ax+b}\,dx = \frac{k}{a}\int \frac{a}{ax+b}\,dx = \frac{k}{a}\cdot \log|ax+b| + k$$

II° CASO – $D(x) = ax^2 + bx + c \qquad \Delta > 0$

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{1}{a}\cdot\left[\int \frac{A}{x-x_1}dx + \int \frac{B}{x-x_2}dx\right] = \frac{A}{a}\cdot \log|x-x_1| + \frac{B}{a}\cdot\log|x-x_2| + k \qquad \begin{cases} A+B=p \\ x_1B + x_2A = -q \end{cases}$$

III° CASO – $D(x) = ax^2 + bx + c \qquad \Delta = 0$

Se $p = 0 \quad \Rightarrow \quad \int \frac{px+q}{ax^2+bx+c}\,dx = \frac{1}{a}\cdot\left[\int \frac{q}{(x-x_1)^2}dx\right] = \frac{q}{a}\cdot\int (x-x_1)^{-2}dx = \frac{q}{a}\cdot\frac{(x-x_1)^{-2+1}}{-2+1} + k$

Se $p \neq 0 \quad \Rightarrow$ Occorre, moltiplicando e dividendo, aggiungendo e sottraendo opportune quantità, trasformare il numeratore nella derivata del denominatore più una costante.

IV° CASO – $D(x) = ax^2 + bx + c \qquad \Delta < 0$

Se $p = 0 \quad \Rightarrow \quad \int \frac{q}{ax^2+bx+c}\,dx = \frac{q}{an^2}\int \frac{1}{\left(\frac{x-m}{n}\right)^2+1}\,dx = \frac{q}{an^2}\cdot n\cdot\int \frac{\frac{1}{n}}{\left(\frac{x-m}{n}\right)^2+1}\,dx = \frac{q}{an}\text{arctg}\,\frac{x-m}{n} + k$

Infatti: dette $x_{1,2} = m \mp n\,i$. *Si ottiene:* $ax^2 + bx + c = a\cdot(x-m-n\,i)\cdot(x-m+n\,i) =$

$$= a\cdot\left[(x-m)^2 - i^2n^2\right] = a\cdot\left[(x-m)^2 + n^2\right] = an^2\cdot\left[\left(\frac{x-m}{n}\right)^2 + 1\right].$$

Se $p \neq 0 \quad \Rightarrow$ Occorre, moltiplicando e dividendo, aggiungendo e sottraendo opportune quantità, trasformare il numeratore nella derivata del denominatore più una costante.

Esistono formule risolutive che permettono di calcolare in modo veloce i precedenti integrali

$$\int \frac{1}{ax^2+bx+c}\,dx = \begin{cases} \dfrac{1}{\sqrt{\Delta}} \cdot \left[log\ |x-x_2| - log\ |x-x_1| \right] + k & se\ \Delta > 0 \\ -\dfrac{1}{a\cdot(x-x_1)} + k & se\ \Delta = 0 \\ \dfrac{2}{\sqrt{-\Delta}} \cdot arctg\ \dfrac{2ax+b}{\sqrt{-\Delta}} + k & se\ \Delta < 0 \\ oppure\ \dfrac{1}{a\cdot n} \cdot arctg\ \dfrac{x-m}{n} + k & con\ x_{1,2} = m \pm n\cdot i \end{cases}$$

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \begin{cases} \dfrac{A}{a} log\ |x-x_1| + \dfrac{B}{a} log\ |x-x_2| + k & se\ \Delta > 0 \\ con \begin{cases} A = \dfrac{px_1+q}{x_1-x_2} \\ B = \dfrac{px_2+q}{x_2-x_1} \end{cases} & \\ \dfrac{p}{a} \cdot log|x-x_1| - \dfrac{px_1+q}{a\cdot(x-x_1)} + k & se\ \Delta = 0 \\ \dfrac{p}{2a} \cdot log\ |ax^2+bx+c| + \dfrac{2aq-bp}{a\cdot\sqrt{-\Delta}} \cdot arctg\ \dfrac{2ax+b}{\sqrt{-\Delta}} + k & se\ \Delta < 0 \end{cases}$$

Esempi del I° caso

$$\boxed{\int \frac{3}{2x-5}\ dx}$$

Soluzione

$$\int \frac{3}{2x-5}\ dx \ = \ \frac{3}{2}\int \frac{2}{2x-5}\ dx \ = \ \frac{3}{2}\cdot log\ |2x-5| \ + \ k \ .$$

Esempi del II° caso $\Delta > 0$

$$\boxed{\int \frac{1}{3x^2-5x-2}\ dx}$$

Soluzione a

Risolvendo $3x^2-5x-2=0$ *si ha:* $\Delta = b^2-4ac = (-5)^2-4\cdot 3\cdot(-2) = 49$

$$x_{1,2} = \frac{-b\mp\sqrt{\Delta}}{2a} \ = \ \frac{5\mp\sqrt{49}}{2\cdot 3} \ = \ \frac{5\mp 7}{6} \ = \ \begin{matrix} x_1 = -\dfrac{1}{3} \\ x_2 = 2 \end{matrix}$$

$$\int \frac{1}{3x^2-5x-2}\ dx \ = \ \int \frac{1}{3\cdot\left(x+\frac{1}{3}\right)\cdot(x-2)}\ dx \ = \ \frac{1}{3}\cdot\int \frac{1}{\left(x+\frac{1}{3}\right)\cdot(x-2)}\ dx \ =$$

$$\frac{1}{\left(x+\frac{1}{3}\right)\cdot(x-2)} \ = \ \frac{A}{\left(x+\frac{1}{3}\right)}+\frac{B}{(x-2)} \ = \ \frac{A\cdot(x-2)+B\cdot\left(x+\frac{1}{3}\right)}{\left(x+\frac{1}{3}\right)\cdot(x-2)}\ .$$

Da cui: $1 = A\cdot(x-2)+B\cdot\left(x+\frac{1}{3}\right);\quad 1 = Ax-2A+Bx+\frac{1}{3}B\ ;\quad 1=(A+B)\cdot x-2A+\frac{1}{3}B$

Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases} A+B=0 \\ -2A+\frac{1}{3}B=1 \end{cases} \quad \begin{cases} A=-B \\ \_\_\_ \end{cases} \quad \begin{cases} \_\_\_ \\ -2\cdot(-B)+\frac{1}{3}B=1 \end{cases} \quad \begin{cases} \_\_\_ \\ 2B+\frac{1}{3}B=1 \end{cases} \quad \begin{cases} \_\_\_ \\ \frac{7}{3}B=1 \end{cases} \quad \begin{cases} \_\_\_ \\ B=\frac{3}{7} \end{cases} \quad \begin{cases} A=-\frac{3}{7} \\ B=\frac{3}{7} \end{cases}$$

Pertanto: $\displaystyle\int \frac{1}{3x^2-5x-2}\ dx \ = \frac{1}{3}\cdot\int \frac{1}{\left(x+\frac{1}{3}\right)\cdot(x-2)}\ dx \ = \ \frac{1}{3}\cdot\int\left[\frac{-\frac{3}{7}}{\left(x+\frac{1}{3}\right)}+\frac{\frac{3}{7}}{(x-2)}\right]dx \ =$

$$= \ \frac{1}{3}\cdot\frac{3}{7}\int\left[\frac{-1}{\left(x+\frac{1}{3}\right)}+\frac{1}{(x-2)}\right]dx \ = \ \frac{1}{7}\int\left[\frac{-1}{\left(x+\frac{1}{3}\right)}+\frac{1}{(x-2)}\right]dx \ = \ \frac{1}{7}\left[log|x-2|-log\left|x+\frac{1}{3}\right|\right]+k\ .$$

Soluzione b

Applicando la formula: $\displaystyle\int \frac{1}{ax^2+bx+c}\ dx \ = \ \frac{1}{\sqrt{\Delta}}\cdot\left[log\ |x-x_2|-log\ |x-x_1|\right] \ + \ k\ .$

$$\int \frac{1}{3x^2-5x-2}\ dx \ = \ \frac{1}{\sqrt{49}}\cdot\left[log\ |x-2|-log\ \left|x+\frac{1}{3}\right|\right] \ + \ k \ = \ \frac{1}{7}\cdot\left[log\ |x-2|-log\ \left|x+\frac{1}{3}\right|\right] \ + \ k\ .$$

$$\boxed{\int \frac{2x+3}{3x^2-4x+1}\,dx}$$

Soluzione a

Risolvende $3x^2-4x+1=0$ *si ha:* $\frac{\Delta}{4}=\left(\frac{b}{2}\right)^2-ac=2^2-3\cdot 1=1>0$

$$x_{1,2}=\frac{-\frac{b}{2}\mp\sqrt{\frac{\Delta}{4}}}{a} = \frac{2\mp\sqrt{1}}{3} = \frac{2\mp 1}{3} = \begin{matrix} x_1=\frac{1}{3} \\ x_2=1 \end{matrix}$$

$$\int \frac{2x+3}{3x^2-4x+1}\,dx = \int \frac{2x+3}{3\cdot\left(x-\frac{1}{3}\right)\cdot(x-1)}\,dx = \frac{1}{3}\cdot\int \frac{2x+3}{\left(x-\frac{1}{3}\right)\cdot(x-1)}\,dx\ .$$

$$\frac{2x+3}{\left(x-\frac{1}{3}\right)\cdot(x-1)} = \frac{A}{\left(x-\frac{1}{3}\right)}+\frac{B}{(x-1)} = \frac{A\cdot(x-1)+B\cdot\left(x-\frac{1}{3}\right)}{\left(x-\frac{1}{3}\right)\cdot(x-1)}\ .$$

Da cui: $2x+3=A\cdot(x-1)+B\cdot\left(x-\frac{1}{3}\right)$; $2x+3=Ax-A+Bx-\frac{1}{3}B$; $2x+3=(A+B)\cdot x-A-\frac{1}{3}B$.

Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases} A+B=2 \\ -A-\frac{1}{3}B=3 \end{cases} \quad \begin{cases} A=2-B \\ \_\_\_ \end{cases} \quad \begin{cases} \_\_\_ \\ -(2-B)-\frac{1}{3}B=3 \end{cases} \quad \begin{cases} \_\_\_ \\ B-\frac{1}{3}B=3+2 \end{cases} \quad \begin{cases} \_\_\_ \\ \frac{2}{3}B=5 \end{cases} \quad \begin{cases} \_\_\_ \\ B=\frac{15}{2} \end{cases} \quad \begin{cases} A=2-\frac{15}{2}=-\frac{11}{2} \\ B=\frac{15}{2} \end{cases}$$

Pertanto l'integrale vale: $\frac{1}{3}\cdot\int \frac{2x+3}{\left(x-\frac{1}{3}\right)\cdot(x-1)}\,dx = \frac{1}{3}\cdot\int\left[\frac{-\frac{11}{2}}{\left(x-\frac{1}{3}\right)}+\frac{\frac{15}{2}}{(x-1)}\right]dx =$

$$= \frac{1}{3}\cdot\left[-\frac{11}{2}\cdot\int\frac{1}{x-\frac{1}{3}}\,dx + \frac{15}{2}\cdot\int\frac{1}{x-1}\,dx\right] = -\frac{11}{6}\cdot\log\left|x-\frac{1}{3}\right| + \frac{5}{2}\cdot\log|x-1| + k$$

Soluzione b

$$\int\frac{px+q}{ax^2+bx+c}\,dx = \frac{A}{a}\log|x-x_1|+\frac{B}{a}\log|x-x_2| + k \qquad \text{con} \quad \begin{cases} A=\frac{px_1+q}{x_1-x_2}=\frac{2\cdot\frac{1}{3}+3}{\frac{1}{3}-1}=\frac{\frac{2}{3}+3}{-\frac{2}{3}}=-\frac{11}{3}\cdot\frac{3}{2}=-\frac{11}{2} \\ B=\frac{q+px_2}{x_2-x_1}=\frac{3+2\cdot 1}{1-\frac{1}{3}}=\frac{5}{\frac{2}{3}}=5\cdot\frac{3}{2}=\frac{15}{2} \end{cases}$$

Pertanto l'integrale è uguale a: $\int\frac{2x+3}{3x^2-4x+1}\,dx = \frac{-\frac{11}{2}}{3}\log\left|x-\frac{1}{3}\right|+\frac{\frac{15}{2}}{3}\log|x-1| + k =$

$$= -\frac{11}{6}\cdot\log\left|x-\frac{1}{3}\right| + \frac{5}{2}\cdot\log|x-1| + k\ .$$

Esempi del III° caso $\Delta = 0$

$$\boxed{\int \frac{1}{4x^2 - 4x + 1}\, dx}$$

Soluzione a

Risolvendo $4x^2 - 4x + 1 = 0$ *si ha:* $\frac{\Delta}{4} = \left(\frac{b}{2}\right)^2 - ac = (-2)^2 - 4 \cdot 1 = 0$. $\quad x_{1,2} = \frac{-\frac{b}{2}}{a} = \frac{2}{4} = \frac{1}{2}$

$$\int \frac{1}{4x^2 - 4x + 1}\, dx = \int \frac{1}{(2x-1)^2}\, dx = \int (2x-1)^{-2}\, dx = \frac{1}{2}\int 2 \cdot (2x-1)^{-2}\, dx =$$

$$= \frac{1}{2} \cdot \frac{(2x-1)^{-2+1}}{-2+1} + k = \frac{1}{2} \cdot \frac{(2x-1)^{-1}}{-1} + k = -\frac{1}{2} \cdot \frac{1}{2x-1} + k = \frac{1}{2-4x} + k$$

Soluzione b

Applicando la formula: $\int \frac{1}{ax^2 + bx + c}\, dx = -\frac{1}{a \cdot (x - x_1)} + k$

$$\int \frac{1}{4x^2 - 4x + 1}\, dx = -\frac{1}{4 \cdot \left(x - \frac{1}{2}\right)} + k = -\frac{1}{4x-2} + k = \frac{1}{2-4x} + k\ .$$

$$\boxed{\int \frac{2x-3}{4x^2-4x+1}\,dx}$$

Soluzione a

Risolvendo $4x^2-4x+1=0$ si ha: $\frac{\Delta}{4}=\left(\frac{b}{2}\right)^2-ac=(-2)^2-4\cdot 1=0$. $x_{1,2}=\frac{-\frac{b}{2}}{a}=\frac{2}{4}=\frac{1}{2}$

$$\int \frac{2x-3}{4x^2-4x+1}\,dx = \int \frac{2x-3}{4\cdot\left(x-\frac{1}{2}\right)^2}\,dx = \frac{1}{4}\cdot\int \frac{2x-3}{\left(x-\frac{1}{2}\right)^2}\,dx$$

$$\frac{2x-3}{\left(x-\frac{1}{2}\right)^2} = \frac{A}{\left(x-\frac{1}{2}\right)}+\frac{B}{\left(x-\frac{1}{2}\right)^2} = \frac{A\cdot\left(x-\frac{1}{2}\right)+B}{\left(x-\frac{1}{2}\right)^2}.$$

Da cui: $2x-3=A\cdot\left(x-\frac{1}{2}\right)+B$; $2x-3=Ax-\frac{1}{2}A+B$; $2x-3=Ax-\frac{1}{2}A+B$.

Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases}A=2\\ -\frac{1}{2}A+B=-3\end{cases}\qquad \begin{cases}\_\_\_\\ -\frac{1}{2}\cdot 2+B=-3\end{cases}\qquad \begin{cases}\_\_\_\\ -1+B=-3\end{cases}\qquad \begin{cases}\_\_\_\\ B=-2\end{cases}\qquad \begin{cases}a=2\\ B=-2\end{cases}$$

Pertanto l'integrale vale: $\frac{1}{4}\cdot\int \frac{2x-3}{\left(x-\frac{1}{2}\right)^2}\,dx = \frac{1}{4}\cdot\int\left[\frac{2}{\left(x-\frac{1}{2}\right)}+\frac{-2}{\left(x-\frac{1}{2}\right)^2}\right]dx =$

$$= \frac{1}{4}\cdot\left[2\cdot\int\frac{1}{x-\frac{1}{2}}\,dx - 2\cdot\int\frac{1}{\left(x-\frac{1}{2}\right)^2}\,dx\right] = \frac{1}{4}\cdot\left[2\cdot\int\frac{1}{x-\frac{1}{2}}\,dx - 2\cdot\int\left(x-\frac{1}{2}\right)^{-2}\,dx\right] =$$

$$= \frac{1}{4}\cdot\left[2\cdot\log\left|x-\frac{1}{2}\right| - 2\cdot\frac{\left(x-\frac{1}{2}\right)^{-2+1}}{-2+1}\right] + k = \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right| - 2\cdot\frac{\left(x-\frac{1}{2}\right)^{-1}}{-1} + k =$$

$$= \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right| + 2\cdot\frac{1}{x-\frac{1}{2}} + k = \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right| + 2\cdot\frac{1}{\frac{2x-1}{2}} + k = \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right|+\frac{1}{2x-1} + k\ .$$

Soluzione b

Applicando la formula: $\int\frac{px+q}{ax^2+bx+c}\,dx = \frac{p}{a}\cdot\log|x-x_1|-\frac{px_1+q}{a\cdot(x-x_1)} + k$

$$\int \frac{2x-3}{4x^2-4x+1}\,dx = \frac{2}{4}\cdot\log\left|x-\frac{1}{2}\right|-\frac{2\cdot\frac{1}{2}-3}{4\cdot\left(x-\frac{1}{2}\right)} + k = \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right|-\frac{-2}{4\cdot\left(x-\frac{1}{2}\right)} + k =$$

$$= \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right|+\frac{1}{2\cdot\left(x-\frac{1}{2}\right)} + k = \frac{1}{2}\cdot\log\left|x-\frac{1}{2}\right|+\frac{1}{2x-1} + k\ .$$

Esempi del IV° caso $\Delta < 0$

$$\boxed{\int \frac{1}{4x^2+8x+7}\,dx}$$

Soluzione a

Trasformando il trinomio in un quadrato di binomio più una costante:

$$4x^2+8x+7 = 4\cdot\left(x^2+2x+\frac{7}{4}\right) = 4\cdot\left(x^2+2\cdot 1\cdot x+\frac{7}{4}+1-1\right) = 4\cdot\left[(x+1)^2+\frac{3}{4}\right] =$$

$$4\cdot\frac{3}{4}\cdot\left[\frac{(x+1)^2}{\frac{3}{4}}+1\right] = 3\cdot\left[\left(\frac{x+1}{\frac{\sqrt{3}}{2}}\right)^2+1\right] = 3\cdot\left[\left(\frac{2x+2}{\sqrt{3}}\right)^2+1\right] .$$

Pertanto: $\int \frac{1}{4x^2+8x+7}\,dx = \int \frac{1}{3\cdot\left[\left(\frac{2x+2}{\sqrt{3}}\right)^2+1\right]}\,dx = \frac{1}{3}\cdot\int \frac{1}{\left(\frac{2x+2}{\sqrt{3}}\right)^2+1}\,dx =$

$$= \frac{1}{3}\cdot\frac{\sqrt{3}}{2}\cdot\int \frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x+2}{\sqrt{3}}\right)^2+1}\,dx = \frac{\sqrt{3}}{6}\cdot arctg\,\frac{2x+2}{\sqrt{3}} + k\,.$$

Soluzione b

Risolvendo $4x^2+8x+7=0$ *si ha:* $\frac{\Delta}{4} = \left(\frac{b}{2}\right)^2 - ac = 4^2-4\cdot 7 = -12$.

$$x_{1,2} = \frac{-\frac{b}{2}\mp\sqrt{\frac{\Delta}{4}}}{a} = \frac{-4\mp\sqrt{-12}}{4} = \frac{-4\mp 2\sqrt{3}i}{4} = -1\mp\frac{\sqrt{3}}{2}i\,. \quad \text{Quindi: } m=-1\,; \quad n=\frac{\sqrt{3}}{2}.$$

$$4x^2+8x+7=0 = 4\cdot\left[x-\left(-1-\frac{\sqrt{3}}{2}i\right)\right]\cdot\left[x-\left(-1+\frac{\sqrt{3}}{2}i\right)\right] =$$

$$= 4\cdot\left[x+1+\frac{\sqrt{3}}{2}i\right]\cdot\left[x+1-\frac{\sqrt{3}}{2}i\right] = 4\cdot\left((x+1)+\frac{\sqrt{3}}{2}i\right)\cdot\left((x+1)-\frac{\sqrt{3}}{2}i\right) =$$

$$= 4\cdot\left((x+1)^2-\left(\frac{\sqrt{3}}{2}i\right)^2\right) = 4\cdot\left((x+1)^2-\frac{3}{4}i^2\right) = 4\cdot\left((x+1)^2+\frac{3}{4}\right) = 4\cdot\frac{3}{4}\cdot\left(\frac{(x+1)^2}{\frac{3}{4}}+1\right) =$$

$$= 3\cdot\left(\frac{4(x+1)^2}{3}+1\right). = 3\cdot\left[\left(\frac{2\cdot(x+1)}{\sqrt{3}}\right)^2+1\right] = 3\cdot\left[\left(\frac{2x+2}{\sqrt{3}}\right)^2+1\right] .$$

Pertanto l'integrale: $\int \frac{1}{4x^2+8x+7}\,dx = \int \frac{1}{3\cdot\left[\left(\frac{2x+2}{\sqrt{3}}\right)^2+1\right]}\,dx = \frac{1}{3}\cdot\int \frac{1}{\left(\frac{2x+2}{\sqrt{3}}\right)^2+1}\,dx =$

$$= \frac{1}{3}\cdot\frac{\sqrt{3}}{2}\cdot\int \frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x+2}{\sqrt{3}}\right)^2+1}\,dx = \frac{\sqrt{3}}{6}\cdot arctg\,\frac{2x+2}{\sqrt{3}} + k\,.$$

Soluzione c

Essendo $m=-1$ *e* $n=\frac{\sqrt{3}}{2}$. *Applicando la formula:* $\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a\cdot n}\cdot arctg\frac{x-m}{n} + k$

$$\int \frac{1}{4x^2+8x+7}\,dx = \frac{1}{4\cdot\frac{\sqrt{3}}{2}}\cdot arctg\frac{x-(-1)}{\frac{\sqrt{3}}{2}} + k = \frac{1}{2\sqrt{3}}arctg\frac{2}{\sqrt{3}}\cdot(x+1) + k = \frac{\sqrt{3}}{6}arctg\frac{2x+2}{\sqrt{3}} + k$$

Soluzione d

Applicando la II[a] Formula: $\int \frac{1}{ax^2+bx+c}\,dx = \frac{2}{\sqrt{-\Delta}}\cdot arctg\frac{2ax+b}{\sqrt{-\Delta}} + k$

$$\int \frac{1}{4x^2+8x+7}\,dx = \frac{2}{\sqrt{48}}\cdot arctg\frac{2\cdot 4\cdot x+8}{\sqrt{48}} + k = \frac{2}{4\sqrt{3}}\cdot arctg\frac{8x+8}{4\sqrt{3}} + k =$$

$$\frac{1}{2\sqrt{3}}\cdot arctg\frac{2x+2}{\sqrt{3}} + k = \frac{\sqrt{3}}{6}\cdot arctg\frac{2x+2}{\sqrt{3}} + k\ .$$

$$\boxed{\int \frac{2x-3}{5x^2-6x+5}\,dx}$$

<u>Soluzione a</u>

Occorre, moltiplicando e dividendo, aggiungendo e sottraendo opportune quantità, trasformare il numeratore nella derivata del denominatore più una costante.

$$\int \frac{2x-3}{5x^2-6x+5}dx = \frac{1}{5}\cdot\int \frac{5\cdot(2x-3)}{5x^2-6x+5}dx = \frac{1}{5}\cdot\int \frac{10x-15}{5x^2-6x+5}dx = \frac{1}{5}\cdot\int \frac{10x-6-9}{5x^2-6x+5}dx =$$

$$= \frac{1}{5}\cdot\left[\int \frac{10x-6}{5x^2-6x+5}dx - 9\cdot\int \frac{1}{5x^2-6x+5}dx\right] .$$

Il primo integrale è immediato. Il secondo invece rientra in uno dei casi precedentemente studiati.

Infatti, trasformando il trinomio in un quadrato di binomio più una costante:

$$5x^2-6x+5 = 5\cdot\left(x^2-\frac{6}{5}x+1\right) = 5\cdot\left(x^2-2\cdot\frac{3}{5}\cdot x+1\right) = 5\cdot\left(x^2-2\cdot\frac{3}{5}\cdot x+\frac{9}{25}-\frac{9}{25}+1\right) =$$

$$5\cdot\left[\left(x-\frac{3}{5}\right)^2+\frac{16}{25}\right] = 5\cdot\frac{16}{25}\cdot\left[\frac{\left(\frac{5x-3}{5}\right)^2}{\frac{16}{25}}+1\right] = \frac{16}{5}\cdot\left[\left(\frac{\frac{5x-3}{5}}{\frac{4}{5}}\right)^2+1\right] = \frac{16}{5}\cdot\left[\left(\frac{5x-3}{4}\right)^2+1\right] =$$

Pertanto l'integrale:

$$\int \frac{1}{5x^2-6x+5}\,dx = \int \frac{1}{\frac{16}{5}\cdot\left[\left(\frac{5x-3}{4}\right)^2+1\right]}\,dx = \frac{5}{16}\cdot\frac{4}{5}\cdot\int \frac{\frac{5}{4}}{\left(\frac{5x-3}{4}\right)^2+1}\,dx = \frac{1}{4}\cdot arctg\,\frac{5x-3}{4} + k_1$$

In definitiva:

$$\int \frac{2x-3}{5x^2-6x+5}\,dx = \frac{1}{5}\cdot\left[\int \frac{10x-6}{5x^2-6x+5}dx - 9\cdot\int \frac{1}{5x^2-6x+5}dx\right] =$$

$$= \frac{1}{5}\cdot\log\left|5x^2-6x+5\right| - \frac{9}{5}\cdot\frac{1}{4}\cdot arctg\,\frac{5x-3}{4} + k$$

$$= \frac{1}{5}\cdot\log\left|5x^2-6x+5\right| - \frac{9}{20}\cdot arctg\,\frac{5x-3}{4} + k \; .$$

<u>*Soluzione b*</u>

il polinomio $\dfrac{2x-3}{5x^2-6x+5} = \dfrac{A\cdot(10x-6)+B}{5x^2-6x+5}$, *dove* $10x-6$ *è la derivata del Denominatore.*

Pertanto si ha: $\dfrac{2x-3}{5x^2-6x+5} = \dfrac{A\cdot(10x-6)+B}{5x^2-6x+5}$ *Da cui:* $2x-3 = A\cdot(10x-6)+B$ *;*

$2x-3 = 10Ax-6A+B$ *Dovendo sussistere l'uguaglianza, deve risultare:*

$$\begin{cases}10A=2\\-6A+B=-3\end{cases}\quad \begin{cases}A=\dfrac{1}{5}\\ \text{\_\_\_\_}\end{cases}\quad \begin{cases}\text{\_\_\_\_}\\-6\cdot\dfrac{1}{5}+B=-3\end{cases}\quad \begin{cases}\text{\_\_\_\_}\\B=-3+\dfrac{6}{5}=-\dfrac{9}{5}\end{cases}\quad \begin{cases}A=\dfrac{1}{5}\\B=-\dfrac{9}{5}\end{cases}$$

Pertanto l'integrale vale:

$$\int\frac{2x-3}{5x^2-6x+5}\,dx = \int\frac{\frac{1}{5}\cdot(10x-6)-\frac{9}{5}}{5x^2-6x+5}\,dx = \frac{1}{5}\cdot\int\frac{10x-6}{5x^2-6x+5}\,dx - \frac{9}{5}\cdot\int\frac{1}{5x^2-6x+5}\,dx\ .$$

Il primo integrale è immediato. Il secondo invece rientra in uno dei casi precedentemente studiati.

Infatti l'integrale: $\displaystyle\int\frac{1}{5x^2-6x+5}\,dx$ *può essere calcolato scomponendo la funzione integranda nel seguente modo:*

Risolvendo $5x^2-6x+5=0$ *si ha:* $\dfrac{\Delta}{4}=\left(\dfrac{b}{2}\right)^2-ac=3^2-5\cdot5=-16$

$$x_{1,2}=\frac{-\frac{b}{2}\mp\sqrt{\frac{\Delta}{4}}}{a} = \frac{3\mp\sqrt{-16}}{5} = \frac{-3\mp4i}{5} = \frac{3}{5}\mp\frac{4}{5}i\ . \qquad \text{Quindi: } m=-\frac{3}{5};\quad n=\frac{4}{5}.$$

$$\text{Pertanto } 5x^2-6x+5 = 5\cdot\left[x-\left(\frac{3}{5}-\frac{4}{5}i\right)\right]\cdot\left[x-\left(\frac{3}{5}+\frac{4}{5}i\right)\right] =$$

$$= 5\cdot\left[x-\frac{3}{5}+\frac{4}{5}i\right]\cdot\left[x-\frac{3}{5}-\frac{4}{5}i\right] = 5\cdot\left[\left(x-\frac{3}{5}\right)+\frac{4}{5}i\right]\cdot\left[\left(x-\frac{3}{5}\right)-\frac{4}{5}i\right] =$$

$$= 5\cdot\left[\left(x-\frac{3}{5}\right)^2-\left(\frac{4}{5}i\right)^2\right] = 5\cdot\left[\left(x-\frac{3}{5}\right)^2+\frac{16}{25}\right] = 5\cdot\frac{16}{25}\cdot\left[\frac{\left(x-\frac{3}{5}\right)^2}{\frac{16}{25}}+1\right] =$$

$$= \frac{16}{5}\cdot\left[\left(\frac{5\cdot\left(x-\frac{3}{5}\right)}{4}\right)^2+1\right] = \frac{16}{5}\cdot\left[\left(\frac{5x-3}{4}\right)^2+1\right].$$

Pertanto l'integrale:

$$\int\frac{1}{5x^2-6x+5}\,dx = \int\frac{1}{\frac{16}{5}\cdot\left[\left(\frac{5x-3}{4}\right)^2+1\right]}\,dx = \frac{5}{16}\cdot\frac{4}{5}\int\frac{\frac{5}{4}}{\left(\frac{5x-3}{4}\right)^2+1}\,dx = \frac{1}{4}\cdot arctg\ \frac{5x-3}{4} + k_1\ .$$

In definitiva:

$$\int\frac{2x-3}{5x^2-6x+5}\,dx = \frac{1}{5}\cdot\int\frac{10x-6}{5x^2-6x+5}\,dx - \frac{9}{5}\cdot\int\frac{1}{5x^2-6x+5}\,dx =$$

$$\frac{1}{5}\cdot\log\left|5x^2-6x+5\right| - \frac{9}{5}\cdot\frac{1}{4}\cdot arctg\ \frac{5x-3}{4} + k = \frac{1}{5}\cdot\log\left|5x^2-6x+5\right| - \frac{9}{20}\cdot arctg\frac{5x-3}{4} + k\ .$$

Soluzione c

$$\int \frac{2x-3}{5x^2-6x+5}\,dx$$

Risolvendo $5x^2-6x+5=0$ *si ha:* $\Delta = b^2-4ac = 6^2-4\cdot 5\cdot 5 = -64$

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{6\pm\sqrt{-64}}{2\cdot 5} = \frac{6\pm 8i}{10} = \frac{3}{5}\pm\frac{4}{5}i.$$ *Quindi:* $m=-\frac{3}{5}$; $n=\frac{4}{5}$.

Applicando la formula:

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{p}{2a}\cdot\log\left|ax^2+bx+c\right| + \frac{2aq-bp}{a\cdot\sqrt{-\Delta}}\cdot \operatorname{arctg}\frac{2ax+b}{\sqrt{-\Delta}} + k$$

$$\int \frac{2x-3}{5x^2-6x+5}\,dx = \frac{2}{2\cdot 5}\cdot\log\left|5x^2-6x+5\right| + \frac{2\cdot 5\cdot(-3)-(-6)\cdot 2}{5\cdot\sqrt{64}}\cdot\operatorname{arctg}\frac{2\cdot 5x-6}{\sqrt{64}} + k =$$

$$= \frac{1}{5}\cdot\log\left|5x^2-6x+5\right| + \frac{-30+12}{40}\cdot\operatorname{arctg}\frac{10x-6}{8} + k =$$

$$= \frac{1}{5}\cdot\log\left|5x^2-6x+5\right| - \frac{18}{40}\cdot\operatorname{arctg}\frac{10x-6}{8} + k =$$

$$= \frac{1}{5}\cdot\log\left|5x^2-6x+5\right| - \frac{9}{20}\cdot\operatorname{arctg}\frac{5x-3}{4} + k\ .$$