Esercizi di Riepilogo

1. $\int \frac{1}{\sqrt{x^2+a^2}}dx$ Moltiplicando e dividendo per $x+\sqrt{x^2+a^2}$ si ha:

$$= \int \frac{x+\sqrt{x^2+a^2}}{\left(x+\sqrt{x^2+a^2}\right)\cdot\sqrt{x^2+a^2}}dx = \int \frac{\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}dx = \int \frac{\frac{x}{\sqrt{x^2+a^2}}+1}{x+\sqrt{x^2+a^2}}dx =$$

$$= \int \frac{1+\frac{2x}{2\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}dx$$ Essendo la funzione del Numeratore la derivata della funzione del denominatore, si può applicare la regola: $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c$.

Pertanto si ha: $\log\left|x+\sqrt{x^2+a^2}\right|+c$

2. $$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx = \frac{1}{2}\int \frac{2\cdot e^{2x}}{\sqrt{1-\left(e^{2x}\right)^2}}dx = \frac{1}{2}\,\text{arcsen}\; e^{2x}+c$$

3. $\int \frac{3x^2+4x-5}{\sqrt[5]{x^3}}dx$ Trasformando $\sqrt[5]{x^3}$ in $x^{\frac{3}{5}}$ si ha: $\int \frac{3x^2+4x-5}{x^{\frac{3}{5}}}dx =$

$$3\int x^{2-\frac{3}{5}}dx+4\int x^{1-\frac{3}{5}}dx-5\int x^{-\frac{3}{5}}dx = 3\int x^{\frac{7}{5}}dx+4\int x^{\frac{2}{5}}dx-5\int x^{-\frac{3}{5}}dx =$$

$$= 3\cdot\frac{5}{12}x^{\frac{12}{5}}+4\cdot\frac{5}{7}x^{\frac{7}{5}}-5\cdot\frac{5}{2}x^{\frac{2}{5}} = \frac{5}{4}\sqrt[5]{x^{12}}+\frac{20}{7}\sqrt[5]{x^7}-\frac{25}{2}\sqrt[5]{x^2} =$$

$$= \frac{5}{4}x^2\cdot\sqrt[5]{x^2}+\frac{20}{7}x\cdot\sqrt[5]{x^2}-\frac{25}{2}\sqrt[5]{x^2} = \sqrt[5]{x^2}\cdot\left[\frac{5}{4}x^2+\frac{20}{7}x-\frac{25}{2}\right].$$

4. $\int \frac{1}{x^2+4x+5}\,dx$ La funzione $x^2+4x+5=1+(x+2)^2$. Pertanto si ha:

$$= \int \frac{1}{1+(x+2)^2}\,dx = \text{arctg}\;(x+2)+c$$

5. $\displaystyle\int \frac{1}{6x^2+x-1}\,dx$ Risolvendo $6x^2+x-1=0$ si ha: $\Delta = b^2-4ac = 1^2-4\cdot 6\cdot(-1) = 25$

$$x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{5\pm\sqrt{49}}{2\cdot 3} = \frac{-1\pm 5}{12} = \begin{array}{l} x_1 = -\frac{1}{2} \\ x_2 = \frac{1}{3} \end{array}$$

Applicando la formula si ottiene:

$$= \frac{1}{\sqrt{\Delta}}\cdot\left[\log\,(x-x_2)-\log\,(x-x_1)\right] + k = \frac{1}{\sqrt{25}}\cdot\left[\log\left(x-\frac{1}{3}\right)-\log\left(x+\frac{1}{2}\right)\right] + k =$$

$$= \frac{1}{5}\cdot\left[\log\left(x-\frac{1}{3}\right)-\log\left(x+\frac{1}{2}\right)\right] + k$$

6. $\displaystyle\int \frac{1}{2x^2-5x+3}\,dx$ Risolvendo $2x^2-5x+3=0$ si ha: $\Delta = b^2-4ac = (-5)^2-4\cdot 2\cdot 3 = 1$.

$$x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{5\pm\sqrt{1}}{2\cdot 2} = \frac{5\pm 1}{4} = \begin{array}{l} x_1 = 1 \\ x_2 = \frac{3}{2} \end{array}$$

Applicando la formula si ottiene:

$$= \frac{1}{\sqrt{\Delta}}\cdot\left[\log\,(x-x_2)-\log\,(x-x_1)\right] + k = \frac{1}{\sqrt{1}}\cdot\left[\log\left(x-\frac{3}{2}\right)-\log\,(x-1)\right] + k =$$

$$= \log\left(x-\frac{3}{2}\right)-\log\,(x-1) + k$$

7. $\displaystyle\int \frac{5}{4x^2+8x+7}\,dx = 5\cdot\int\frac{1}{4x^2+8x+7}\,dx$

Risolvendo $4x^2+8x+7=0$ si ha: $\Delta = b^2-4ac = 8^2-4\cdot 4\cdot 7 = -48$.

$$x_{1,2} = \frac{-\frac{b}{2}\pm\sqrt{\frac{\Delta}{4}}}{a} = \frac{-4\pm\sqrt{-12}}{4} = \frac{-4\pm 2\sqrt{-3}}{4} = \frac{-4\pm 2\sqrt{3}i}{4} = -\frac{4}{4}\pm\frac{2\sqrt{3}i}{4} = -1\pm\frac{\sqrt{3}}{2}i\,.$$

Da cui si ha: $m=-1$; $n=\frac{\sqrt{3}}{2}$.

L'integrale $\displaystyle\int\frac{1}{4x^2+8x+7}\,dx$ applicando la formula: $\displaystyle\frac{1}{a\cdot n}\,arctg\,\frac{x-m}{n} + k$ vale pertanto:

$$= \frac{1}{4\cdot\frac{\sqrt{3}}{2}}\,arctg\,\frac{x-(-1)}{\frac{\sqrt{3}}{2}} + k = \frac{1}{2\sqrt{3}}\,arctg\,\frac{2}{\sqrt{3}}\cdot(x+1) + k = \frac{\sqrt{3}}{6}\,arctg\,\frac{2x+2}{\sqrt{3}} + k\,.$$

In definitiva si ha: $\displaystyle 5\cdot\int\frac{1}{4x^2+8x+7}\,dx = \frac{5\sqrt{3}}{6}\,arctg\,\frac{2x+2}{\sqrt{3}} + k$

Mimmo Corrado

8. $\int \frac{3x-5}{4x^2+8x+7}\,dx$

Risolvendo $4x^2+8x+7=0$ si ha: $\Delta = b^2-4ac=8^2-4\cdot4\cdot7=-48$.

$$x_{1,2}=\frac{-\frac{b}{2}\pm\sqrt{\frac{\Delta}{4}}}{a} = \frac{-4\pm\sqrt{-12}}{4} = \frac{-4\pm2\sqrt{-3}}{4} = \frac{-4\pm2\sqrt{3}i}{4} = -\frac{4}{4}\pm\frac{2\sqrt{3}i}{4} = -1\pm\frac{\sqrt{3}}{2}i \,.$$

Occorre fare in modo che a numeratore compaia la derivata del denominatore:

$$= \int \frac{3\cdot\left(x-\frac{5}{3}\right)}{4x^2+8x+7}\,dx = 3\cdot\int\frac{x-\frac{5}{3}}{4x^2+8x+7}\,dx =$$

$$= \frac{3}{8}\cdot\int\frac{8\cdot\left(x-\frac{5}{3}\right)}{4x^2+8x+7}\,dx = \frac{3}{8}\cdot\int\frac{8x-\frac{40}{3}}{4x^2+8x+7}\,dx = \frac{3}{8}\cdot\int\frac{8x-\frac{40}{3}+8-8}{4x^2+8x+7}\,dx =$$

$$= \frac{3}{8}\cdot\int\frac{8x+8-\frac{64}{3}}{4x^2+8x+7}\,dx = \frac{3}{8}\left[\int\frac{8x+8}{4x^2+8x+7}\,dx-\frac{64}{3}\int\frac{1}{4x^2+8x+7}\,dx\right] =$$

$$= \frac{3}{8}\cdot\left[\log\left|4x^2+8x+7\right|-\frac{64}{3}\cdot\frac{1}{4\cdot\frac{\sqrt{3}}{2}}arctg\frac{2x+2}{\sqrt{3}}\right]+k =$$

$$= \frac{3}{8}\cdot\left[\log\left|4x^2+8x+7\right|-\frac{64}{3}\cdot\frac{\sqrt{3}}{6}arctg\frac{2x+2}{\sqrt{3}}\right]+k =$$

$$= \frac{3}{8}\cdot\log\left|4x^2+8x+7\right|-\frac{4\sqrt{3}}{3}arctg\frac{2x+2}{\sqrt{3}}+k\,.$$

9. $\int \frac{2x-1}{x^2-5x+6}\,dx$

Risolvendo $x^2-5x+6=0$ si ha: $\Delta = b^2-4ac = 1$; $x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a} = \begin{matrix}x_1=2\\ x_2=3\end{matrix}$

La funzione integranda si può scrivere come: $\frac{2x-1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$; $\begin{cases}A=-3\\B=5\end{cases}$ Da cui:

$$\int\frac{2x-1}{x^2-5x+6}\,dx = \int\frac{-3}{x-2}\,dx+\int\frac{5}{x-3}\,dx = -3\cdot\log|x-2| + 5\cdot\log|x-3| + k\,.$$

10. $\int \frac{3}{4x^2+8x+7}\,dx$

Risolvendo $4x^2+8x+7=0$ si ha: $\Delta = b^2-4ac = -48$; $x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a} = -1\pm\frac{\sqrt{3}}{2}i$

Utilizzando la formula: $\frac{1}{a\cdot n}arctg\frac{x-m}{n} + k$ si ha:

$$= \int\frac{3}{4x^2+8x+7}\,dx = 3\cdot\int\frac{1}{4x^2+8x+7}\,dx = 3\cdot\frac{1}{4\cdot\frac{\sqrt{3}}{2}}arctg\frac{x+1}{\frac{\sqrt{3}}{2}} + k =$$

$$= \frac{3}{2\sqrt{3}}arctg\frac{2x+2}{\sqrt{3}} + k = \frac{\sqrt{3}}{2}arctg\frac{2x+2}{\sqrt{3}} + k$$

Mimmo Corrado

11. $\displaystyle\int \frac{x^2+1}{4x^2-8x+9}\,dx$

Calcoliamo l'integrale: $\displaystyle\int \frac{x^2+1}{4x^2-8x+9}\,dx$ eseguendo la divisione:

$$\begin{array}{cccc|l} x^2 & & +1 & & 4x^2-8x+9 \\ \hline -x^2 & +2x & -\frac{9}{4} & & \frac{1}{4} \\ \hline & +2x & -\frac{5}{4} & & \end{array}$$

$$=\int \frac{1}{4}\,dx+\int \frac{2x-\frac{5}{4}}{4x^2-8x+9}\,dx$$

$$= \int \frac{1}{4}\,dx+\int \frac{\frac{8x-5}{4}}{4x^2-8x+9}\,dx \quad = \quad \int \frac{1}{4}\,dx+\frac{1}{4}\int \frac{8x-5}{4x^2-8x+9}\,dx \quad (\#).$$

Calcoliamo pertanto l'integrale: $\displaystyle\int \frac{8x-5}{4x^2-8x+9}\,dx$ Risolvendo $4x^2-8x+9=0$ si ha: $\Delta=-80$;

$$x_{1,2}=\frac{-\frac{b}{2}\pm\sqrt{\frac{\Delta}{4}}}{a} = \frac{4\pm\sqrt{-20}}{4} = \frac{4\pm 2\cdot\sqrt{5}\cdot i}{4} = 1\pm\frac{\sqrt{5}}{2}i$$

Facciamo comparire a numeratore la derivata del denominatore, aggiungendo e togliendo 3:

$$\int \frac{8x-5}{4x^2-8x+9}\,dx = \int \frac{8x-5-3+3}{4x^2-8x+9}\,dx = \int \frac{8x-8}{4x^2-8x+9}\,dx+\int \frac{3}{4x^2-8x+9}\,dx =$$

$$= \int \frac{8x-8}{4x^2-8x+9}\,dx+3\cdot\int \frac{1}{4x^2-8x+9}\,dx =$$

Il primo integrale è immediato. Il secondo si applica la formula: $\displaystyle\frac{1}{a\cdot n}\,arctg\,\frac{x-m}{n} + k$.

$$= \log\left|4x^2-8x+9\right|+3\cdot\frac{1}{4\cdot\frac{\sqrt{5}}{2}}\,arctg\,\frac{x-1}{\frac{\sqrt{5}}{2}} + k =$$

$$= \log\left|4x^2-8x+9\right|+\frac{3}{2\sqrt{5}}\,arctg\,\frac{2\cdot(x-1)}{\sqrt{5}} + k =$$

$$= \log\left|4x^2-8x+9\right|+\frac{3\sqrt{5}}{10}\,arctg\,\frac{2x-2}{\sqrt{5}} + k.$$ In definitiva, da (#) si ha:

$$\int \frac{1}{4}\,dx+\frac{1}{4}\int \frac{8x-5}{4x^2-8x+9}\,dx = \frac{1}{4}x+\frac{1}{4}\log\left|4x^2-8x+9\right|+\frac{1}{4}\cdot\frac{3\sqrt{5}}{10}\,arctg\,\frac{2x-2}{\sqrt{5}} + k =$$

$$= \frac{1}{4}x+\frac{1}{4}\log\left|4x^2-8x+9\right|+\frac{3\sqrt{5}}{40}\,arctg\,\frac{2x-2}{\sqrt{5}} + k.$$

Mimmo Corrado

12. $\displaystyle\int \frac{x^4-3}{x^2+2x+1}\,dx$ Eseguendo la divisione si ottiene:

$$
\begin{array}{ccccc|l}
x^4 & & & & -3 & x^2+2x+1 \\ \cline{6-6}
-x^4 & -2x^3 & -x^2 & & & x^2-2x+3 \\ \cline{1-5}
= & -2x^3 & -x^2 & & -3 & \\
 & 2x^3 & +4x^2 & +2x & & \\ \cline{2-5}
 & = & 3x^2 & +2x & -3 & \\ \cline{3-5}
 & & -3x^2 & -6x & -3 & \\ \cline{3-5}
 & & = & -4x & -6 &
\end{array}
$$

$$= \int \left(x^2-2x+3\right)dx - \int \frac{4x+6}{x^2+2x+1}\,dx$$

$$= \int \left(x^2-2x+3\right)dx - \int \frac{4x+6}{x^2+2x+1}\,dx \quad = \quad \int \left(x^2-2x+3\right)dx - 2\int \frac{2x+3}{x^2+2x+1}\,dx \quad (\#).$$

Calcoliamo pertanto l'integrale: $\displaystyle\int \frac{2x+3}{x^2+2x+1}\,dx$ Risolvendo $4x^2-8x+9=0$ si ha: $\Delta=0$;

$x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-2 \pm \sqrt{0}}{2 \cdot 1} = -1$. Si ottiene: = $\displaystyle\int \frac{2x+3}{(x+1)^2}\,dx$

il polinomio $\dfrac{2x+3}{(x+1)^2}$ può essere riscritto come somma di due funzioni:

$$\frac{2x+3}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} = \frac{A\cdot(x+1)+B}{(x+1)^2}.$$

Da cui: $2x+3 = A\cdot(x+1)+B$; $2x+3 = Ax+A+B$;

Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases} A=2 \\ A+B=3 \end{cases} \qquad \begin{cases} \_\_\_ \\ 2+B=3 \end{cases} \qquad \begin{cases} \_\_\_ \\ B=1 \end{cases} \qquad \begin{cases} A=2 \\ B=1 \end{cases}$$

Pertanto l'integrale vale: $\displaystyle\int \frac{2x+3}{x^2+2x+1}\,dx = \int \frac{2x+3}{(x+1)^2}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx =$

$$= 2\cdot\int \frac{1}{x+1}\,dx + \int (x+1)^{-2}dx = 2\cdot \log\,|x+1| + \frac{(x+1)^{-2+1}}{-2+1} + k =$$

$$= 2\cdot\log\,|x+1| - \frac{1}{x+1} + k.$$ In definitiva da (#) si ha:

$$\int \left(x^2-2x+3\right)dx - 2\int \frac{2x+3}{x^2+2x+1}\,dx = \frac{x^3}{3} - x^2 + 3x - 2\cdot\left[2\cdot\log\,|x+1| - \frac{1}{x+1}\right] + k =$$

$$= \frac{x^3}{3} - x^2 + 3x - 4\cdot\log\,|x+1| + \frac{2}{x+1} + k.$$

Mimmo Corrado

13. $\displaystyle\int \frac{x^3+1}{x^2+7x+12}\,dx$ Eseguendo la divisione si ottiene:

$$\begin{array}{rrrr|l}
x^3 & & & +1 & x^2+7x+12 \\ \cline{5-5}
-x^3 & -7x^2 & -12x & & x-7 \\ \cline{1-4}
 & -7x^2 & -12x & +1 & \\ \cline{2-4}
 & +7x^2 & +49x & +84 & \\ \cline{2-4}
 & & +37x & +85 &
\end{array}$$

$$= \int (x-7)\,dx + \int \frac{37x+85}{x^2+7x+12}\,dx$$

Calcoliamo pertanto l'integrale: $\displaystyle\int \frac{37x+85}{x^2+7x+12}\,dx$

Risolvendo $x^2+7x+12=0$ si ha: $x_{1,2} = \dfrac{-7\pm\sqrt{49-48}}{2} = \dfrac{-7\pm1}{2} = \left\langle \begin{array}{l} x_1=-4 \\ x_2=-3 \end{array}\right.$.

il polinomio $\dfrac{37x+85}{x^2+7x+12}$ può essere riscritto come somma di due funzioni:

$$\frac{37x+85}{x^2+7x+12} = \frac{A}{(x+3)} + \frac{B}{(x+4)} = \frac{A\cdot(x+4)+B\cdot(x+3)}{(x+3)\cdot(x+4)} = \frac{Ax+4A+Bx+3B}{(x+3)\cdot(x+4)} = \frac{(A+B)\cdot x+4A+3B}{(x+3)\cdot(x+4)}$$

Da cui: $37x+85=(A+B)\cdot x+4A+3B$. Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases} A+B=37 \\ 4A+3B=85 \end{cases} \quad \begin{cases} B=37-A \\ 4A+3\cdot(37-A)=85 \end{cases} \quad \begin{cases} - \\ 4A+111-3A=85 \end{cases} \quad \begin{cases} - \\ A=85-111 \end{cases} \quad \begin{cases} B=+63 \\ A=-26 \end{cases}$$

Pertanto l'integrale vale:

$$\int \frac{37x+85}{x^2+7x+12}\,dx = \int \frac{-26}{x+3}\,dx + \int \frac{63}{x+4}\,dx = -26\cdot\log|x+3| + 63\cdot\log|x+4| + k$$

In definitiva:

$$\int \frac{x^3+1}{x^2+7x+12}\,dx = \int (x-7)\,dx + \int \frac{37x+85}{x^2+7x+12}\,dx = \frac{x^2}{2} - 7x - 26\cdot\log|x+3| + 63\cdot\log|x+4| + k$$

Mimmo Corrado

14. $\displaystyle\int \frac{3x-2}{x^2-2x+2}\,dx$

Risolvendo $x^2-2x+2=0$ si ha: $x_{1,2}=1\pm\sqrt{1-2}=1\pm\sqrt{-1}$; $\Delta<0$.

Il polinomio $\dfrac{3x-2}{x^2-2x+2}$ occorre riscriverlo nella forma: $\dfrac{A\cdot(2x-2)+B}{x^2-2x+2}$,

(dove $2x-2$ rappresenta la derivata del denominatore).

Pertanto si ha: $\dfrac{3x-2}{x^2-2x+2}=\dfrac{A\cdot(2x-2)+B}{x^2-2x+2}$ Da cui: $3x-2=A\cdot(2x-2)+B$; $3x-2=2Ax-2A+B$

Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases}2A=3\\-2A+B=-2\end{cases}\qquad\begin{cases}A=\dfrac{3}{2}\\ \_\_\_\end{cases}\qquad\begin{cases}\_\_\_\\-2\cdot\dfrac{3}{2}+B=-2\end{cases}\qquad\begin{cases}\_\_\_\\B=-2+3\end{cases}\qquad\begin{cases}A=\dfrac{3}{2}\\B=1\end{cases}$$

Pertanto: $\displaystyle\int\frac{3x-2}{x^2-2x+2}\,dx=\int\frac{\frac{3}{2}\cdot(2x-4)+1}{x^2-2x+2}\,dx=\frac{3}{2}\cdot\int\frac{2x-4}{x^2-2x+2}\,dx+\int\frac{1}{x^2-2x+2}\,dx$

Il primo integrale è immediato *(essendo il numeratore la derivata del denominatore)*.

Il secondo invece rientra in uno dei casi precedentemente studiati. Infatti l'integrale può essere calcolato scomponendo la funzione integranda nel seguente modo:

$x^2-2x+2=(x^2-2x+1)+1=1+(x-1)^2$. Pertanto si ha:

$$\int\frac{1}{x^2-2x+2}\,dx=\int\frac{1}{1+(x-1)^2}\,dx=arctg\ (x-1)+c$$

In definitiva:

$$\int\frac{3x-2}{x^2-2x+2}\,dx=\frac{3}{2}\cdot\int\frac{2x-4}{x^2-2x+2}\,dx+\int\frac{1}{x^2-2x+2}\,dx=$$

$$=\frac{3}{2}\cdot\log\left|x^2-2x+2\right|+arctg\ (x-1)+k\ .$$

Mimmo Corrado

15. $\displaystyle\int \frac{2x^3-5x^2-x+16}{x^2-4x+5}\,dx$ Eseguendo la divisione si ottiene:

$$\begin{array}{rrrr|l} 2x^3 & -5x^2 & -x & +16 & x^2-4x+5 \\ \cline{5-5} -2x^3 & +8x^2 & -10x & & 2x+3 \\ \cline{1-4} & +3x^2 & -11x & +16 & \\ & -3x^2 & +12x & -15 & \\ \cline{2-4} & & x & +1 & \end{array}$$

$$= \int (2x+3)\,dx + \int \frac{x+1}{x^2-4x+5}\,dx$$

Calcoliamo pertanto l'integrale: $\displaystyle\int \frac{x+1}{x^2-4x+5}\,dx$

Risolvendo $x^2-4x+5=0$ si ha: $x_{1,2}=2\pm\sqrt{4-5}=2\pm\sqrt{-1}$; $\Delta<0$.

Il polinomio $\dfrac{x+1}{x^2-4x+5}$ occorre riscriverlo nella forma: $\dfrac{A\cdot(2x-4)+B}{x^2-4x+5}$,

(dove $2x-4$ rappresenta la derivata del denominatore).

Pertanto si ha: $\dfrac{x+1}{x^2-4x+5} = \dfrac{A\cdot(2x-4)+B}{x^2-4x+5}$ Da cui: $x+1=A\cdot(2x-4)+B$; $x+1=2Ax-4A+B$.

Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases} 2A=1 \\ -4A+B=1 \end{cases} \qquad \begin{cases} A=\dfrac{1}{2} \\ \_\_\_\_ \end{cases} \qquad \begin{cases} \_\_\_\_ \\ -4\cdot\dfrac{1}{2}+B=1 \end{cases} \qquad \begin{cases} \_\_\_\_ \\ B=1+2 \end{cases} \qquad \begin{cases} A=\dfrac{1}{2} \\ B=3 \end{cases}$$

Pertanto: $\displaystyle\int \frac{x+1}{x^2-4x+5}\,dx = \int \frac{\frac{1}{2}\cdot(2x-4)+3}{x^2-4x+5}\,dx = \frac{1}{2}\cdot\int \frac{2x-4}{x^2-4x+5}\,dx + 3\cdot\int \frac{1}{x^2-4x+5}\,dx$

Il primo integrale è immediato *(essendo il numeratore la derivata del denominatore)*.

Il secondo invece rientra in uno dei casi precedentemente studiati. Infatti l'integrale può essere calcolato scomponendo la funzione integranda nel seguente modo:

$x^2-4x+5 = (x^2-4x+4)+1 = 1+(x-2)^2$. Pertanto si ha:

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{1+(x-2)^2}\,dx = arctg\ (x-2)+c$$

In definitiva:

$$\int \frac{2x^3-5x^2-x+16}{x^2-4x+5}\,dx = \int (2x+3)\,dx + \int \frac{x+1}{x^2-4x+5}\,dx =$$

$$= \int (2x+3)\,dx + \frac{1}{2}\cdot\int \frac{2x-4}{x^2-4x+5}\,dx + 3\cdot\int \frac{1}{x^2-4x+5}\,dx =$$

$$= x^2+3x+\frac{1}{2}\cdot log\ \left|x^2-4x+5\right| + 3\cdot arctg\ (x-2) + k\ .$$

Mimmo Corrado

16. $\displaystyle\int \frac{x^2+1}{6x-9x^2}\,dx$ Eseguendo la divisione si ottiene:

$$
\begin{array}{cccc|c}
x^2 & & +1 & & -9x^2+6x \\
\cline{5-5}
-x^2 & +\frac{2}{3}x & +1 & & -\frac{1}{9} \\
\cline{1-4}
 & +\frac{2}{3}x & +1 & &
\end{array}
$$

$$= \int -\frac{1}{9}\,dx + \int \frac{\frac{2}{3}x+1}{-9x^2+6x}\,dx$$

Calcoliamo pertanto l'integrale: $\displaystyle\int \frac{\frac{2}{3}x+1}{-9x^2+6x}\,dx$

Risolvendo $-9x^2+6x=0$ si ha: $x\cdot(-9x+6)=0$.

il polinomio $\displaystyle\frac{\frac{2}{3}x+1}{-9x^2+6x}$ può essere riscritto come somma di due funzioni:

$$\frac{\frac{2}{3}x+1}{-9x^2+6x} = \frac{A}{x}+\frac{B}{-9x+6} = \frac{-9Ax+6A+Bx}{x\cdot(-9x+6)} = \frac{(-9A+B)\cdot x+6A}{x\cdot(-9x+6)}$$

Da cui: $\frac{2}{3}x+1=(-9A+B)\cdot x+6A$. Dovendo sussistere l'uguaglianza, deve risultare:

$$\begin{cases}-9A+B=\frac{2}{3}\\ 6A=1\end{cases} \quad \begin{cases}-9\cdot\frac{1}{6}+B=\frac{2}{3}\\ A=\frac{1}{6}\end{cases} \quad \begin{cases}-\frac{3}{2}+B=\frac{2}{3}\\ -\end{cases} \quad \begin{cases}B=\frac{2}{3}+\frac{3}{2}\\ -\end{cases} \quad \begin{cases}B=\frac{4+9}{6}\\ -\end{cases} \begin{cases}B=\frac{13}{6}\\ A=\frac{1}{6}\end{cases}$$

Pertanto l'integrale vale:

$$\int \frac{\frac{2}{3}x+1}{-9x^2+6x}\,dx = \int \frac{\frac{1}{6}}{x}\,dx + \int \frac{\frac{13}{6}}{-9x+6}\,dx = \frac{1}{6}\int \frac{1}{x}\,dx + \frac{13}{6}\int \frac{1}{-3\cdot(3x-2)}\,dx = \frac{1}{6}\int \frac{1}{x}\,dx - \frac{13}{18}\int \frac{1}{3x-2}\,dx$$

$$= \frac{1}{6}\int \frac{1}{x}\,dx - \frac{13}{18}\cdot\frac{1}{3}\int \frac{3}{3x-2}\,dx = \frac{1}{6}\log|x| - \frac{13}{54}\log|3x-2| \;+\; k$$

In definitiva:

$$\int \frac{x^2+1}{6x-9x^2}\,dx = \int -\frac{1}{9}\,dx + \int \frac{\frac{2}{3}x+1}{-9x^2+6x}\,dx = -\frac{1}{9}x+\frac{1}{6}\log|x| - \frac{13}{54}\log|3x-2| \;+\; k\ .$$

Mimmo Corrado

17. $\displaystyle\int \frac{x^5 + 3x^2}{x^2 + 1}\, dx$ Eseguendo la divisione si ottiene:

$$
\begin{array}{ccccc|l}
x^5 & & +3x^2 & & & x^2+1 \\
-x^5 & -x^3 & & & & \overline{x^3 - x + 3} \\
\hline
 & -x^3 & +3x^2 & & & \\
 & +x^3 & & +x & & \\
\hline
 & & +3x^2 & +x & & \\
 & & -3x^2 & & +3 & \\
\hline
 & & & +x & +3 &
\end{array}
$$

$$\int \left(x^3 - x + 3\right) dx - \int \frac{x+3}{x^2+1}\, dx$$

Il primo integrale è immediato: $\displaystyle\int \left(x^3 - x + 3\right) dx = \frac{x^4}{4} - \frac{x^2}{2} + 3x + k$

Il secondo integrale si decompone in:

$$\int \frac{x+3}{x^2+1}\, dx = \int \frac{x}{x^2+1}\, dx + \int \frac{3}{x^2+1}\, dx = \frac{1}{2} \cdot \int \frac{2x}{x^2+1}\, dx + 3\int \frac{1}{x^2+1}\, dx =$$

$$= \frac{1}{2} \cdot log\left|x^2 + 1\right| + 3 \cdot arctg\ x + k$$

In definitiva:

$$\int \frac{x^5 + 3x^2}{x^2+1}\, dx = \int \left(x^3 - x + 3\right) dx - \int \frac{x+3}{x^2+1}\, dx = \frac{x^4}{4} - \frac{x^2}{2} + 3x + \frac{1}{2} \cdot log\left|x^2 + 1\right| + 3 \cdot arctg\ x + k$$

Mimmo Corrado

18. $\displaystyle\int_0^{\pi} \frac{\cos x}{1+sen^2 x}\, dx$

Determiniamo innanzitutto l'integrale indefinito: $\displaystyle\int \frac{\cos x}{1+sen^2 x}\, dx$

applicando la regola di integrazione: $\displaystyle\int \frac{f'(x)}{1+[f(x)]^2}\, dx = arctg\ f(x) + k$.

Si ha: $\displaystyle\int \frac{\cos x}{1+sen^2 x}\, dx = arctg(senx) + k$

E quindi: $\displaystyle\int_0^{\pi} \frac{\cos x}{1+sen^2 x}\, dx = [arctg(senx)]_0^{\pi} = [arctg(sen\pi) - arctg(sen0)] = [arctg\ 0 - arctg\ 0] = 0$.

19. $\displaystyle\int_0^{1} (-x+3)\cdot e^{-x}\, dx$

Determiniamo innanzitutto l'integrale indefinito: $\displaystyle\int (-x+3)\cdot e^{-x} dx$

applicando la regola di integrazione per parti: $\displaystyle\int f'(x)\cdot g(x)\, dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\, dx$.

Considerando $\begin{cases} f'(x) = e^{-x} & fattore\ differenziale \\ g(x) = (-x+3) & fattore\ finito \end{cases} \Rightarrow \begin{cases} f(x) = -e^{-x} \\ g'(x) = -1 \end{cases}$ si ha:

$$\int (-x+3)\cdot e^{-x} dx = -(x-3)\cdot\left(-e^{-x}\right) - \int (-1)\cdot\left(-e^{-x}\right) dx = (x-3)\cdot e^{-x} - \int e^{-x}\, dx =$$

$$= (x-3)\cdot e^{-x} + e^{-x} + k = x\cdot e^{-x} + 3\cdot e^{-x} + e^{-x} + k = (x-2)\cdot e^{-x} + k$$

E quindi: $\displaystyle\int_0^1 (-x+3)\cdot e^{-x}\, dx = \left[(x-2)\cdot e^{-x}\right]_0^1 = \left[(1-2)\cdot e^{-1} - (0-2)\cdot e^{-0}\right] = \left[-e^{-1} + 2\cdot 1\right] = 2 - \frac{1}{e}$.

20. $\displaystyle\int_{\frac{1}{2}}^{\frac{e}{2}} \frac{\log 2x}{x}\, dx$

Determiniamo innanzitutto l'integrale indefinito: $\displaystyle\int \frac{\log 2x}{x}\, dx$

Applicando la regola di integrazione: $\displaystyle\int [f(x)]^n \cdot f'(x)\, dx = \frac{1}{n+1}\cdot f(x)^{n+1} + k$ si ha:

$$\int \frac{\log 2x}{x}\, dx \quad \int [\log 2x]^1 \cdot \frac{1}{x}\, dx = \frac{1}{1+1}[\log|2x|]^{1+1} + k = \frac{1}{2}\log^2|2x| + k$$

E quindi: $\displaystyle\int_{\frac{1}{2}}^{\frac{e}{2}} \frac{\log 2x}{x}\, dx = \left[\frac{1}{2}\log^2|2x|\right]_{\frac{1}{2}}^{\frac{e}{2}} = \left[\frac{1}{2}\log^2 e - \frac{1}{2}\log^2 1\right] = \frac{1}{2}\log^2 e = \frac{1}{2}\cdot 1 = \frac{1}{2}$.

Mimmo Corrado

21. $\displaystyle\int_0^1 \frac{arctg\ x}{1+x^2}\ dx$

Determiniamo innanzitutto l'integrale indefinito: $\displaystyle\int \frac{arctg\ x}{1+x^2}\ dx$

Applicando la regola di integrazione: $\displaystyle\int [f(x)]^n \cdot f^I(x)\ dx = \frac{1}{n+1} \cdot f(x)^{n+1}\ +\ k$ si ha:

$$\int \frac{arctg\ x}{1+x^2}\ dx\ = \int [arctg\ x]^1 \cdot \frac{1}{1+x^2}\ dx\ = \frac{1}{1+1} arctg^{1+1} x\ +\ k\ = \frac{1}{2} arctg^2 x\ +\ k$$

E quindi: $\displaystyle\int_0^1 \frac{arctg\ x}{1+x^2}\ dx\ = \left[\frac{1}{2} arctg^2 x\right]_0^1 = \left[\frac{1}{2} arctg^2 1 - \frac{1}{2} arctg^2 0\right] = \frac{1}{2} \cdot \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{32}$.

Mimmo Corrado